

Das wohltemperirte Clavier

Tuning and Musical Structure By

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In memoriam Friedrich Smend 1893-1980

1. Resumption and Introduction

Bach's well-tempered tuning, as reconstituted, is a baroque apotheosis of the musical *Tri-Unitas*, resuming my earlier paper¹. This tuning, in technical terms, contains five well-tempered and seven pure fifths, respectively, 5,7. Four equal well-tempered fifths make up the third c-e, and the remaining one, the 'tempering fifth' B-f sharp is virtually of the same size^{2,3}. Hence, including the seven perfect fifths of the temperament, their respective numbers will be 4,1,7. The tuning is based upon the triad of C-major, the enlarged third of which beats at the same rate as its flat fifth. In what follows, sources substantiating C as the centre of tonality will be quoted, and authors, dealing with the symbolism of the musical Trinity, the *trias harmonica perfecta*. Then some musico-numerical features will be taken up, mostly from Bach's well-tempered clavier.

2. The tonal centre C and trinitarian symbolism

From the outset, C-major has no sharps or flats, from the viewpoint of keyboard music. But upon C, the crucial well-tempered C major triad is located which plays a decisive role, and in fact, determines Bach's entire tuning. As early as 1588, Zarlino starts his 12 Modi with C, in his *Istitutioni harmoniche*, see also Zenck⁴. Werckmeister, in *Musicae Mathematicae Hodegus Curiosus*, 1686, allocates to the unisonus 1 the C, being the root which bears the entire tree with all its fruits: '1 ist der unisonus, welchem wir allhier das C zuschreiben wollen/diese 1. ist die Wurtzel/aus welcher der gantze Baum mit seinen Friichten entspringt.' In the same book, on p. 147, Werckmeister recognizes the most perfect triad or Tri-Unisonus within the harmonics 4.5.6., i.e. c-e-g⁵. An early source for the triad as a symbol, is Schneegass, 1591: by virtue of being perfectly consonant, the triad can be considered as 'an appropriate and pleasing symbol of the Trinity', see Dahlhaus⁶. It also

occurs in *Exercitationes musicae duae*, 1600, by the Thomaskantor Sethus Calvisius, or in Joh. Lippius, *Synopsis musicae novae*, 1612; see Zenck⁴, and, more general, Dammann⁷.

3. Musical examples

1. Prelude No. 1 in C major, BWV 846 in its final form. Its extension, apart from the factorization $35 = 5 \times 7$, can also be considered as the juxtaposition of the 3rd and 5th in thoroughbass, making up a triad. But it will be much more significant to follow this prelude's way through to its final form. It appears on p. 84 as Nr. 29 in the *Klavierbuchlein für Anna Magdalena Bach*, where 29 may be related to the monogram JSB or to *Soli Dei Gloria*, SDG, according to Smend⁸. According to NBA V, Vol 4, by von Dadelen⁹, this page 84 is the only one within this booklet which was prepared from the outset with 8 staves for 4 systems, obviously for the express purpose of receiving a version of this prelude. The number of this page is interesting, see my contribution¹⁰ to the EHM and also Currie¹¹ in the *Quarterly Bach*. I have noted that the first bar of the tempering-tonality, B major fugue No. 23, is $1932 = 23 \times 84 = 23 \times 6 \times 14$. As $14 = \text{BACH}$, Fig. 1, under this ordinal number the C major prelude figures in another source, the *Klavierbuchlein für Wilhelm Friedemann Bach* 1720, NBA, see Plath¹². This version, BWV 846a, has 27 bars, *trias trinitatisper multiplicationem*, and is written on sheet 14r. 27 was Bach's ultimate choice for the associated fugue, and extending the prelude to 35 bars. A further version by Forkel of the prelude exists, BGA vol. XIV¹³, and its length is another significant number for the WTC, namely 24. These features observed suggest a distinguished role of this prelude in the formation-phase of the WTC. Its 'Berlin-Autograph' is the authoritative source, Bach's own fair copy in calligraphic writing. Its facsimile cannot be recommended enough¹⁴. See also Dehnhard²⁰. It is likely that such a marvellous

The authentic reading of the C-major prelude's last three bars (BWV 846)



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calligraphic text would contain very few errors or omissions by negligence, in particular, within the first prelude. But attention must be drawn to the tie in bar 33, joining two minims in the bass. Its appearance does not match anything else within this prelude and apparently it was set at a different time from the remainder of the piece. However, many current editions show not only in bar 33, but also in bar 34 just one bass-keystroke, amounting to an implication of even two such ties; a single one would make even less sense. But are whole notes not written much more naturally like O than using a tie d d ? In contrast to current editorial practice, there may be two more authentic key-strokes for the piece, in place of 549, rather a total of $32 \times 16 + 2 \times 17 + 5 = 551$. But this number breaks up nicely as: $551 = 19 \times 29$, admitting an interpretation of 19 as the tuning-steps and 29

1	2	3	4	5	6	7	8
α	β	γ	δ	ε	ζ	η	θ
9	10	11	12	13	14	15	16
ι	κ	λ	μ	ν	ο	π	ρ

17 18 19 2p 21 22 23 H,
r f (u n > ; P ! •

as SDG or JSB. Thus, by numerology, Fig. 3 may be proposed as authentic and at the same time the logical one. I owe to Dr. Walther Dehnard (personal communication) the observation that only later copies from the autograph show this tie; it is absent in copies taken earlier than 1732. 2. **Fugue No. 2** in C minor. Its architecture can be characterized by the symmetrical structure of its entries, employing the parallel major key: 3(C minor) + 1 (E flat major) + 3 (C minor) + Coda See Ratz¹⁵, also Werker⁶ and Hahn¹⁷. Thus, the numbers of subject-entries are 3,1,3 which can be regarded as a triptych, centred on the unique entry of E flat, the 'trinity-key'. In bar 29 (SDG, JSB), the triple C sounds, the centre of tonality, at which point the fugue proper ends and the coda upon the pedal C takes over. Within WTC I, the number of this bar may be interpreted as a trinitarian CREDO, factorizing $129 = 3 \times 43$; via the usual letter-numbering, Smend.⁸ The fugue of 31 bars (trinity-unity) ends at bar 131, another triptych. Thus C major and minor, preludes and fugues, characteristically comprise 131 bars. Similarly, No. 3 C sharp major and No. 4 C sharp minor preludes and fugues comprise, complementary, 313 bars and we recognize the C minor fugue's trintvch. These features will be written out:

prelude and fugue	bars	sum	interpretation
C major and minor	131	5	well-temp. fifths
C sharp major and minor	313	7	pure fifths
	444 = 12 × 37		37 = JCHR; 12 chrom. steps

According to Hahn, 37 means the monogram JCHR, as $37 = 9 + 3 + 8 + 17$. **3. Prelude No. 7** in E flat major, the tonality of Trinity. As stated earlier¹, it commences at bar 577 of WTC I. Its length of 70 bars, Fig. 2, is equivalent to JESUS, see Hahn¹⁷. The key of

Summa	M.70	D394	△612	□1154	○390	△4020	□2428	○20518
I	9	9	45	81	48	105	288	229
E	5	5	15	25	24	35	55	125
S	18	90	171	324	102	1140	2109	5832
U	20	200	210	400	114	1540	2820	8000
S	18	90	171	324	102	1140	2109	5832

Hier ist zu bemerken/daß VI die Zahlbuchstaben in dem Namen lesVs, die erste vollkommene Zahl / von welcher zu sehen in der folgenden XXXIX.

g-minor at its midpoint, bar 35, dividing it by the ratio of the unity, will be discussed later. Hahn writes (1973):

Within the structure, there are 24 appearances of the subject. As much as 35 is the product of 5×7 , their sum is 12. This the composer underlines by partitioning this fugue within the prelude into four parts of 5 and 7, and 7 and 5 entries. These 24 entries of the subject are accompanied by 12 countersubjects, 7 complete ones and 5 varied ones, mostly shortenend. The constant elements of this fugue are therefore: 24 appearances of the subject 12 countersubjects (7 complete, 5 varied) 1 countersubject across all parts (bar 53ff, at 14th entry) 37 fixed elements

The following fugue No. 7, in E flat major, has 37 bars!

The tonality at the midpoint of the prelude, bar 35, is interesting: In Bach's system, within the G minor triad, between its tempered major and minor thirds, the proportion 5:7 holds for their beat-rates, reflecting the numbers of well-tempered and perfect fifths of the system! For the relevant acoustical theory, see Janssens¹⁸. In order to find out whether Bach might have been aware of this relation, one may look up his g-minor contribution to WTC I. Its prelude of 19 bars finishes at bar 1369 of WTC I, and $1369 = 37 \times 37$, so the fugue starts at 1370; unitas and 37 juxtaposed. It ends at 1403, BACH and Trinity, whilst $1403 = 23 \times 61$: 23 belonging to the B major key, and 61 a concatenation from the beat-ratio 6:1.

The length of this fugue No. 16, together with its prelude's bars, amounts to $19 + 34 = 53$ which may be interpreted as the numbers pertaining to fifth and third. Remembering the setup 131/313 right at the beginning of WTC I, let us now consider the entire G-major and minor complex. Its first prelude and fugue, yields $19 + 86 = 105$, by

gematria equivalent to *Trinitas*. This number 105 = $3 \times 5 \times 7$ Bach makes use of, for example: in bar 27 = $3 \times 3 \times 3$ of the D major prelude a *third* voice enters as the unison tenor on a (A=1) of the pedal, on the 105th keystroke of the bass (Hahn, p. 109). Another example for 105 can be noted from the fourth duet's original edition (See NBA IV/4 and Krit. Ber. p.45 by M. Tessler). There, with the second crotchet of bar 79, the alto-clef is being introduced to stay up to the third crotchet of bar 105= TRINITAS. In between, this alto renders 129 = $3 \times$ CREDO keystrokes, see also Smend⁸. But are there any other trinitarian allusions? The number of bars with alto clef, is 105-78 = 27 and within these entire 27 bars, there are 131 keystrokes, the well-known triptych!

Following Hahn⁷, the G minor fugue, with its theme of 5 + 7 keystrokes, is structured as follows: the 17 entries of the subject occur 5 times in its first half, and 7 times in its second half; the situation is reversed for the countersubject: 7 times within the fugue's first half, and 5 times in the remainder. Additionally the answer occurs 5 times. To conclude, the total bars of G major and minor count 105 + 53 = 158, equivalent to the full name, Johann Sebastian Bach.

4. Fugue No. 3 in C sharp major of WTC II. To the discussion of the corresponding prelude in my preceding paper, it should be noted that the fugue was originally set in C major and had 19 bars length, before it was transposed and extended. See also Zenck⁴.

5. The 14 canons discovered 1975. These canons were edited by Wolff¹⁹ and records are available: Curtis/Haas, and Junghanns/Tracey. The end of the autograph, kept now in Paris, shows an *etc* sign as can be seen in the facsimile of the edition. It is advisable to write this out? The following *lusus ingenii* eats thereby lost:

letter	value	summation	interpretation
E	5		well tempered fifths
T	19	24	tonalities
C	3	27	trinitas, per mult.

or, else, utilizing the juxtaposition ETC = 5193 = 9×577 : trias trinitatis, now per additionem, and the well-known prime 577 related to the closure of the circle of fifths. Pushing numerological mannerism even further, 19 may be considered to be centred within 53; the constituents of the triads 5 and 3, whilst tempering takes 19 steps: 5 (19) 3. If etc. is written out, what remains when *et cetera* is converted, will be 73 + 1 when the final 'a' is added. As is well known, 74/73 is then the superparticular ratio approximating the pythagorean comma, but this *lusus ingenii* is rather far fetched, if compared to the original one.

6. Prelude and fugue No. 23 in B major. Within the first bar, the prelude's soprano has 14 = BACH keystrokes, and there is a total of 19 notes in the first bar, see¹. The prelude commences upon the tonal pedal note B, and the entire associated complex carried by the pedal B, comprises 37 keystrokes, for these 3 voices. Again a correspondence of 37 to J.CHR. may be assumed. Such an opening introduction, in view of the frequent J. J. at the beginning of musical pieces, interpreted to mean *Jesu juva*, is not unusual for baroque composers. But here, in B major, the crucial tempering-procedure is to be applied for tuning. From the alto E sharp of bar 3 onwards, there descends a chain of essentially 6 fifths in the bass: E sharp-A sharp-D sharp-G sharp-C sharp until F sharp-B is reached, the *tempering-fifth*. It occurs at the 11th step of the well-tempering sequence, after six previous perfect fifths to be tuned². The B thus reached, is the 66th keystroke (6×11) of the prelude, simultaneously, the 37th keystroke (tied, suspension) sounds of the treble: may this be interpreted, upon well-tempering, J.CHR. will be above? The tempering-fifth B-F sharp occurs 3 times in the piece: also in bar 13 (trinity-unity) and from bar 14 = BACH to bar 15. At this instance, the treble has rendered 105 = $3 \times 5 \times 7$ = TRINITAS keystrokes and is set off by a semiquaver-rest from its continuation. All this should be verified, if it is not coincidental, as well as the fact that the B-major prelude comprises a total of 417 keystrokes which brings us back to the resumption and introduction, where we had:

4 well-tempered fifths C-G-D-A-E
1 tempering fifth B-F sharp
7 perfect fifths

Also the factorization of the number chosen by Bach for the keystrokes of the B major prelude is remarkable:

$$417 = 3 \times 139$$

fifths Trinity 19: tuning-steps

Here the 19 steps are centred numerologically upon the *Trinitas*.

7. Notes on the 4 duets. The duets, 1739, show similar features as explained above. For example, connecting by multiplication the Trinity 3 with the 19 tuning-steps, will yield 57, the concatenation of the numbers of well-tempered and pure fifths. This number is the extension in keystrokes of the first duet's theme. Or else, dividing the number of duets according to the *unitas*, I + II and III + IV, gives us the following numbers of bars:

$$\begin{array}{ccccccc} & & 14 & & & & \\ & & 73 & + & 149 & + & 39 & + & 108 \\ & & 222 & & & & 147 & & \end{array}$$

147 = set up of pure and tempered fifths,

from the *Uni-Trinitas*

222 = 6×37 (J.CHR.),

6 = *numerus perfectus*

A reasoning similar to the one for the 551 keystrokes of the C major prelude, will confirm a tie from bar 44 to 45 of the first duet's bass, omitted erroneously in the first printing; no autographs are known. Then, there will be 491 keystrokes for the bass of the E minor duet: the trias trinitatis per additionem, 9, centered numerologically within 41 = J.S. BACH.

4. Conclusion and outlook

Studies in this spirit of Bach's music can be a rewarding enrichment. They also illustrate how the composer employs his profoundly spiritual tuning-system, unified with the architecture of his music. But who will be the Champollion for Bach's esoteric language, where in his compositions is his musical stone of Rosetta?

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