

Was Bach a Mathematician ?

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Musica est exercitium arithmeticae occultum nescientis se numerari animi. G. W. Leibniz (1646-1716)

1. Introduction

In a letter to the editor of the *English Harpsichord Magazine* I specified * J. S. Bach's temperament for the 'Forty-Eight', without giving, however, any proof of authenticity. In a subsequent second letter ², the practical method for its implementation on the harpsichord was demonstrated. Nor will such proof be presented here; the intention is rather to shed some light on the question, whether Johann Sebastian

Bach can be considered a mathematician, an aspect which is certainly related to the authenticity of the tuning-system I specified. This contribution is based on the lecture which I delivered on 1 August 1977, at the Bruges 5th International Harpsichord Week held within the framework of the 14th International Fortnight of Music.

Bach's well-tempered tuning for all 24 tonalities follows in an elementary way as the logical and

uniquely-defined outcome of stipulating that the best major triad attainable in a balanced and smooth temperament for all 24 tonalities, should occur therein exactly once ^{3>4}. This triad will rest on C-major, the centre of tonality. Equivalent to this formulation, as a descriptive specification, there will be 5 equal well-tempered fifths and seven pure fifths closing their circle of 7 octaves; where four such flattened fifths make up the best third c-e of the tempered C-major triad, and the remaining well-tempered fifth ascends at B-fff. In Baroque music the major triad was considered a *trias harmonica perfecta* and a symbol of the Trinity. In Bach's temperament the C-major triad is distinguished in that its third c-e beats at the same rate as its fifth c-g, which is the ratio of unity. Thus the *trias harmonica perfecta* on C not only symbolizes the Trinity, but simultaneously the perfection of the 'unitas' in baroque musical theory ^{5>6}. The advantage of this simultaneous mutual adaptation within the triad can be seen as follows: In mean-tone triads the purity of the third is impaired by the beating fifth, and in the triad of equal temperament the rather sharp third interferes with the nearly pure fifth. This demonstrates the soundness and justification for tempering the beat-rates according to the ratio of unity. Within the C-major triad four such well-tempered fifths make up the second octave of the well-tempered third e. In keeping with the principle of the perfection of the 'unitas', there is but one such most perfect triad within the temperament, namely on C-major. Implementing this system on the harpsichord ², the first interval to be tempered is the fifth f#-B after the descent of six pure fifths, starting from middle c (octave-transpositions wherever necessary). B-ffts correctly tempered, as soon as the beat-rates within the B-major triad attain the ratio 6:1, according to the proportion (beat-rate B-dtt) : (beat-rate B-fft) = 6:1. This method obviously holds for any value chosen for the standard pitch because it is formulated via ratios of beat-rates. It must be stressed that apart from octaves and fifths no other tuning-intervals occur in the procedure for laying the bearings ⁷ of this most perfect temperament, which consequently requires a minimal number of exactly 19 steps proper, not counting any redundant checks, but including the one for the circle's closure. 2. Musical mathematics and mathematical music

The well-tempered tuning derives from the triad of C-major, which is a symbol of the Tri-Unity. Its bearings can be laid in exactly 19 steps, starting off at middle c: c'-c⁰ and c'-f⁰. These 19 steps are no more than are required to generate the pythagorean comma via fifths and octaves. As regards the closure of the circle of fifths in Bach's system we have

$$\begin{array}{ccccccc} 5 & + & 7 & = & 7 \\ \text{w.-t. 5ths} & & \text{pure 5ths} & & \text{Octaves} \end{array}$$

Juxtaposing these figures forms the prime number

577, its sum of digits being, of course, 19. In what follows, some specific and rather obvious features within Bach's musical work will now be enumerated, all of which can easily be verified and interpretations will be proposed, to be replaced by better ones as soon as they become available. Most of these examples will be taken from the two books of the '48', or the Well-tempered Clavier (WTC).

1. Prelude and fugue in C-major, WTC I: The prelude's texture is an arpeggio-figuration of 5 parts. The entire prelude counts 35=5x7 bars, this decomposition reflecting the temperament's well-tempered and pure fifths. After bars 1 and 4, the C-major triad reappears in bar 19. In contrast to the first 5 groups of 4 measures each ⁸, a group of three bars, comprising 21, 22, and 23 leads to bar 24, symbolizing the Tri-Unity which permits the usage of all 24 tonalities. This group of three bars is a pronounced rhetoric ellipse ⁹ and was considered so bizarre by Schwencke that he attempted an improvement by proposing to insert a bar after 22, rectifying the diminished third Ftt -Ab in the bass. As regards the fugue, it is 27=3x3x3 bars in length and we recognize the *trias trinitatis per multiplicationem*. The fugue's theme is signed by the composer via 14=BACH keystrokes, using the kabbalistic gematria of numbering letters, common in the Baroque period ¹⁰. Therein the correspondence in equivalence is A=1, B=2, C=3 ... H=8, I=J=9, ... , U=V=20, ... , Z=24. At the end of the fugue, the 14th keystroke of the last, the 27th bar, the c^{'''}, has in the autograph a '27' written at its side, stating explicitly the fugue's extension ¹¹. Similar numbering appears at the end of prelude 4 in Ctt -minor, counting 39=3x13 bars, and in the B-minor Mass ¹² after '*patrem omnipotentem*', 84=6x14 (or 7x12).

2. Prelude 7 in Eb-major; this tonality is usually associated with the Trinity, see also 3 below. The prelude extends over 70 bars, and has an introduction of 9=3+3+3 bars, *trias trinitatis per additionem*. The following structural features are noted:

bar	tonality	remarks	decomposition
24/25	Bb-major	24 tonalities	25=5x5
35	g-minor	midpoint	35=5x7
49	Bb-major	49=7x7	

Thus, the entire form of 70 bars is divided up according to the ratio of unity, at which point the minor parallel of this section's initial Bb-major occurs, namely g-minor. Such features are not uncommon in the architecture of Bach's rich musical forms. To all multiplicative combinations of the figures 5 and 7, there correspond formal-harmonic structures, in bars 5x5, 5x7 and 7x7. In bar 27=3x3x3, two voices merge into the unison upon the entry of the tenor on Eb. If all bars of the WTC I are counted through, the prelude 7 in Eb-major starts at 577, a reminder of the figures upon which Bach bases his closing of the circle of fifths within

his temperament. As a cross-reference, let us consider the prelude in Cft -major of the WTC II. This tonality of prelude 3, WTC II is at the greatest distance from C-major. In these most remote regions the requirements of good tempering for all 24 modes are pushed to the extreme limit, and all the triads on I, IV and V are still acceptable, purely Pythagorean. This prelude extends over a total of 50 measures and its second section starts at bar 25; a numerical division in the ratio of the unity. All the 24 bars of its first section are uniformly structured: The treble and alto parts of the figuration count 14 keystrokes and the lower half of tenor and bass 10, adding up to 24 keystrokes per bar. As there are 24 identically structured bars, the prelude's first section comprises $24 \times 24 = 576$ keystrokes, to be interpreted as a mathematical apotheosis of well-tempering by squaring the number of tonalities accessible. Back to WTC I, Bach terminated the d-minor fugue with this ordinal number of bars, so that the Trinity Eb-major prelude starts with bar 577 of WTC I as we have already noted. In the eft-major II prelude, the 577th keystroke is absorbed within the five-part chord of its bar $25 = 5 \times 5$, involving another squaring-apotheosis, this time using the number of well-tempered fifths. This first section of the C# -major II prelude is built upon a foundation of 48 bass-keystrokes, which equals the total number pertaining to the '48' formed by WTC I and II. One might become curious to know what features in the WTC II leads to its Eb-major prelude of the Trinity, and one finds a number of $27 = 3 \times 3 \times 3$ bars in the preceding d-minor fugue.

3. Clavierübung HI, 1739. Framed by prelude and fugue in Eb-major, the tonality of Trinity, it has 27 movements, just as the New Testament has 27 books, containing the 4 gospels, amongst which no one takes a distinguished place. An analogous constellation can be seen in the Clavierübung III: among its 27 movements there is a particular group, namely the 4 duets. Omitting the distinguished best tonality¹⁸ within Bach's tuning of C-major, those remaining closest to the perfection of pure thirds, are e-minor, F-major, G-major and a-minor, in which we recognize the tonalities of the 4 duets.

4. Prelude 9 in E-major: Its ordinal number itself is the *trias trinitatis per additionem*, $9 = 3 + 3 + 3$. Set throughout in 3 parts, the sole and striking exception is seen in bar $14 = \text{BACH}$, reducing to only 2 voices, the treble having 19, the bass $14 = \text{BACH}$ keystrokes. This place is distinguished as well by the figuration in semiquavers which occurs nowhere else but is necessary to achieve these numbers.

5. Fugue a 3 in A-major, WTC I, Nr. 19: Its ordinal number corresponds to the characteristic number of tuning steps. Further related allusions to tuning can therefore be expected, even more so, because according to the gematria, $A = 1$, and writing music in the form of letters is the intrinsic feature of German organ tablature, see e.g. Marshal¹⁴. In line with the *unitas* of A, the fugue starts with an isolated

eighth-note, followed by three rests, set in measures of $9/8$, alluding to the *trias trinitatis per additionem*. Instead of going into more details, let us note that the first bar of the fugue in A within the WTC is 1577, which can be interpreted as a juxtaposition of the *unitas* 1, with 577, the now well-known complex related to well-tempered tuning, a prime number. In its entirety, its decomposition reads $1577 = 19 \times 83$. Nothing more need to be said here about 19, and apart from other usages, the prime number 83 appears juxtaposed to 14 in the first bar of prelude gtt-minor I, and is the number of bars of the fugue in C-major, WTC II.

6. Fourteen canons were discovered in 1975, handwritten in Bach's personal copy of the Goldberg-variations. This number of canons, $14 = \text{BACH}$, is a rather remarkable baroque signature by the composer to his Clavierübung IV.

7. Prelude and fugue 23 in B-major: Its tonal triad is the crucial one within the 19 tuning-steps of well-tempering. This prelude No. 23 has a length of 19 bars, like those of G-major, No. 15 and g-minor, No. 16. There are 19 keystrokes within the first measure of the B-major prelude. Counting all bars across WTC I, this prelude commences at 1913 and its 19th bar therefore ends at 1931 such that the following juxtaposition of numbers is noted:

Starts	1913 Tuningsteps, Unity, Trinity
Ends	1931 Tuningsteps, Trinity, Unity

Both the sums of digits are $\text{BACH} = 14$: $1 + 9 + 1 + 3 = 1 + 9 + 3 + 1$. Strikingly remarkable, both these numbers are primes. The 23rd fugue thus starts at bar $1932 = 6 \times 14 \times 23$. This factorization contains the beat-ratio 6, $14 = \text{BACH}$, and the ordinal number 23 of this fugue within the WTC I. In order to render these extraordinary factorizations possible, the numbering had to be unambiguous, such that Bach had to abstain from using repetition-forms up to this point. Promptly, at the earliest opportunity possible, we find AABB in the B-minor prelude Nr. 24. As much as on a micro-scale in prelude I the preceding measures lead to 24, the number of tonalities, on a macro-scale Prelude and Fugue 23 lead to the usage of all tonalities. The theme of the fugue, in the tenor, has $14 = \text{BACK}$ keystrokes, cf. fugue I in C-major. Here the tonal answer, alto, deploys over the 14th keystroke a figuration of the B-major triad and it is just via figurations the tempering of the fifth B-fS and the third B-dft is effected, listening alternately to these two intervals and taking up the rhythm of their beats. Apart from the signature of the fugue by the 14 initial tenor-keystrokes up to the entry of the alto, the entire first tenor-phrase counts $41 = \text{J.S. BACH}$ keystrokes, to be considered as a duplicated, further signature. If we look into the fugue's bar 14, at its beginning nothing else appears than the triad of Bach's tonality of Trinity, though in the enharmonic disguise Dft rather than Eb-major. Similarly, as the prelude I of the WTC went through

C-major triads in the bars 1,4,19 a glance at bar 19 of the B-major prelude shows another figuration of the tonal triad. Let us now look at the final bar of the fugue in B-major, in the tonality of which the third B-dtt and the fifth B-fft must beat according to the ratio of 6. As a unique feature, an exception within the WTC, the treble sounds the third. It is useful to check with the facsimile of the autograph¹⁹, the way the chord in the final bar is written, somewhat like this:



This final measure of the fugue is $1965=3 \times 5 \times 131$ and the latter prime number could be looked at as the triptych Unity, Trinity, Unity, in the usual sense of juxtaposition. This view can be substantiated by Blankenburg's trinitarian interpretation¹³ of the rather large number 131 in the Credo of the B-minor mass. The entire factorization contains once more the 3=Trinity and 5, the number of well-tempered fifths.

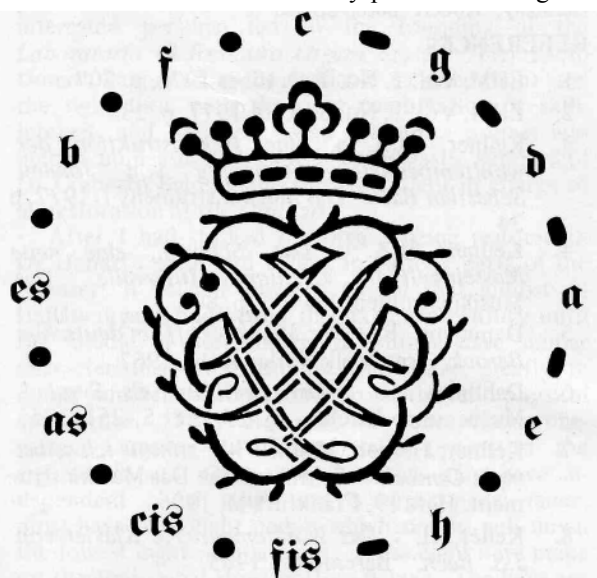
8. Improvements to the Well-tempered Clavier by others, who knew better and had a more highly developed sense of mathematical beauty and order than the composer himself: Schwencke proposed adding one bar to the first prelude in C-major after the original bar 22. More drastic, Busoni even exchanged the two preludes in Eb-major between the two volumes of the '48'. The reader himself may judge these attempts.

3. Conclusion and final remarks

Some evidence has been established, in what way, from a single and one-sided aspect, Bach can be regarded as a mathematician. The extent to which this holds, has gone unnoticed by musicology for nearly 257 years after the WTC I was signed by the composer, but it would be unreasonable to blame musicologists for this fact, or to reproach them, as they usually have no formal mathematical training, nor the mathematical viewpoint^{*5}. However, as regards the authenticity of Bach's tuning, and the rigour inherent in the methods of musicological science, all of what has been said here, provides of course no proof whatsoever to this problem. Establishing nothing more than some rather obvious mathematical allusions cannot possibly be accepted as conclusive proof. Unfortunately, this question of authenticity seems to be the only one in which many

people are interested, in the context of Bach's well-tempered tuning. I expressly refrain from divulging what I consider proofs of authenticity, because after I have formulated this temperament associated with a tuning method previously never heard of, I am quite content to have identified them as those employed by Bach. This system is as well-defined as it is well-tempered and it appears unlikely that anything different could be drawn up to perform with 12 fixed pitches per octave, keyboard music through all 24 tonalities¹⁶, the derivation is quite elementary. Instead of furnishing proofs, where I might be suspected of being keen and biased, I calmly await constructive action from others, either dismissing this temperament as an unauthentic fake, or furnishing the proofs that it belonged to Bach. Such efforts of search are certainly warranted because of the quality in all respects of the music, and its profoundness which deserves to be studied with devotion. I do not consider the proofs of authenticity as a sort of deliverable consumer-goods, but I am pleased to hear that this temperament is now being widely used by musicians to perform music.

Nor is Bach's seal of 1722 any proof—switching for



an instant to graphic art - although this seal occurs for the first time in the very year, the fair copy of the WTC was dated. The crown of Bach's seal is centered with respect to a gothic trifoil, a symbol for the Trinity, the crown of the monogram having a total of 14 jewels, 2 partly covered, a constellation of 5 and 7 gemstones respectively. Going back to baroque numerology, it might be noted that the full name, Johann Sebastian Bach yields upon converting its letters into numbers, and summing up, a value of 158. In turn, the sum of these digits is 14, the further sum of which figures is 5 in the unity of digits, and corresponds to the number of well-

tempered fifths, like the sum of J.S. BACH=41, the inversion or crab of 14. All these operations are common in numerology. Accepting such a numerological equivalence of '5' for BACH, it might be formulated for well-tempered tuning, the ditonic comma should be divided by Bach, as these fifths are flattened by 1/5 of this pythagorean comma, which I call 'Bach-comma'³. Some more numerological features of this temperament may finally be added: Cff, the first of 12 steps ascending after C, has a distance of 19 Bach-commas from that basis. Whereas in musical acoustics considerable thought has been devoted to the 7th natural harmonic *⁷, it is seen that the Trinity Eb in Bach's temperament, the pythagorean minor third of C, coincides rather accurately with the 19th natural harmonic and the number of letters of the composer's full name, Johann Sebastian Bach, is 19.

This seal of Bach, augmented with the circle of fifths according to this interpretation, has been printed in the five colours gold, white, red and black on a green background. Readers who would like to receive such a postcard, should send the author a self-addressed envelope and two international postal reply-coupons; address: D 61 Darmstadt, Germany, Robert Bosch-Strasse 5.

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